# The Cyclical Majority Problem

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**The problem of the cyclical majority is presented and some new, simulated results for 3, 4, 5, ... , 40 issues and 3, 5, 7, ... , 37 judges are reported.**

KEY WORDS AND PHRASES: Arrow's paradox, cyclical majority, simulation, voter's paradox, voting paradox
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## The Problem

We introduce an unsolved combinatorial problem, which has been of recent interest to economists and political scientists. Then we present some new results, which were obtained from a computer simulation.

The problem can be stated in the following scenario. A committee consisting of three judges $A$, $B$, and $C$ wishes to choose, by a simple majority vote, one of three solutions $i$ or $j$ or $k$. The preferences of each of the judges can be tabulated simply as:

Judge $A$ prefers $i$ to $j$ and $j$ to $k$,

Judge $B$ prefers $j$ to $k$ and $k$ to $i$,

Judge $C$ prefers $k$ to $i$ and $i$ to $j$.

If the committee votes on $i$ and $j$, $i$ will win; if the vote is on $j$ and $k$, $j$ will win; if the vote is on $i$ and $k$, $k$ will win. Thus the committee prefers $i$ to $j$, $j$ to $k$, and $k$ to $i$. This committee can agree on no simple majority winner—whatever issue is chosen, two of the members prefer another to it. Such an impasse is called a *cyclical majority*.

The unsolved combinatorial problem is: Given an odd number $m$ of judges each with a transitive preference ordering over $n$ $(\geq 3)$ issues, what is the probability that there will be no clear majority winner of the $n$ issues when the committee votes on the issues in paired comparisons? Alternatively, how many of the $(n!)^m$ possible committee voting outcomes result in a cyclical majority? Three important points should be kept in mind:

(a) each judge has a *strongly ordered* transitive preference ordering over the issues—no judge is ever indifferent to any pair of distinct issues;

(b) each judge is equally likely to have one of the $n!$ possible preference orderings over the $n$ issues as his preference ordering;[1]

(c) a cyclical majority is not said to have occurred when there is one issue that more than half the judges prefer to all others issues even though there may be cycles below first choice. For example, if in the preference pattern shown above there was an issue $l$ that all three judges preferred to $i$, $j$, and $k$, then $l$ would be a clear winner and the cyclical majority has not occurred in spite of the cycle within issues $i$, $j$, and $k$.

This problem has a considerable history which we will not discuss here. The interested reader is referred to Black [2 (1948)], Arrow [1 (1951)], Riker [8 (1961)], Campbell and Tullock, [3 (1965)], and Klahr [6 (1966)]. We can summarize the relevant history by saying that many have tried to find a simple, closed-form expression for the probability of a cyclical majority but none has succeeded.[2] Neither have we succeeded, but in our attempt we devised a method for simulating the voting process to estimate the probabilities which appears much more efficient than one previous simulation, Klahr [6], and more accurate than another, Campbell and Tullock [3].

## The Results

In Table I we show the fractions of a random sample of 15,000 voting patterns that led to a cyclical majority for $m$ judges and $n$ issues, $m = 3, 5, \cdots, 37, n = 3, 4, \cdots, 40$. The fractions are given only to the nearest one-hundredth because, as Klahr [6] has shown, this is the theoretical accuracy limit for the probability when so small a sample is used.

The same sample is used to generate all the entries in the table and these estimates are *not* independent of each other, as we explain below. Any one of the estimates of a cyclical majority from 15,000 patterns is accurate to more than .01

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[1] This assumption is called "the assumption of an impartial culture" by Garman and Kamien [5]. They also discuss the problem of partial cultures.

[2] It is, perhaps, an issue upon which reasonable men can agree to differ whether or not an expression such as $\sum_{\mathbf{r} \in R}$ (simple expression involving the vector $\mathbf{r}$ and the quantities $m$ and $n$) is a simple, closed-form expression, when to enumerate all "$\mathbf{r} \in R$" is currently an impossibly long computation for general $m$ and $n$. Garman and Kamien [5], DeMeyer and Plott [4], and Niemi and Weisberg [7] provide formulae of varying complexity and use various combinatorial structures—primarily vectors, matrices, and multinomial coefficients—each unable to be evaluated for small $m$ and $n$ in any reasonable time.

TABLE 1. PROBABILITY ESTIMATES OF A CYCLICAL MAJORITY. TOTAL SAMPLE, 15,000 VOTES

*(Entries in table are not independent)*

| NUMBER OF ISSUES | NUMBER OF JUDGES | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 | 23 | 25 | 27 | 29 | 31 | 33 | 35 | 37 |
| 3 | .05 | .07 | .07 | .07 | .08 | .08 | .08 | .08 | .08 | .08 | .08 | .08 | .08 | .08 | .08 | .09 | .08 | .09 |
| 4 | .11 | .14 | .14 | .16 | .16 | .16 | .17 | .17 | .16 | .16 | .16 | .16 | .16 | .16 | .17 | .17 | .17 | .17 |
| 5 | .16 | .20 | .21 | .23 | .23 | .23 | .24 | .23 | .24 | .24 | .23 | .24 | .24 | .24 | .24 | .24 | .24 | .24 |
| 6 | .20 | .25 | .27 | .28 | .28 | .29 | .29 | .30 | .30 | .30 | .30 | .30 | .30 | .30 | .31 | .30 | .30 | .31 |
| 7 | .24 | .29 | .32 | .33 | .33 | .34 | .35 | .35 | .35 | .35 | .35 | .36 | .36 | .36 | .36 | .36 | .36 | .36 |
| 8 | .27 | .33 | .36 | .38 | .38 | .38 | .39 | .39 | .40 | .40 | .41 | .41 | .40 | .40 | .41 | .40 | .41 | .41 |
| 9 | .30 | .37 | .39 | .41 | .41 | .42 | .43 | .42 | .43 | .44 | .44 | .44 | .45 | .44 | .45 | .45 | .44 | .45 |
| 10 | .32 | .39 | .43 | .45 | .45 | .45 | .46 | .46 | .46 | .47 | .48 | .47 | .48 | .48 | .48 | .48 | .48 | .48 |
| 11 | .35 | .42 | .45 | .47 | .48 | .49 | .49 | .49 | .49 | .50 | .50 | .50 | .51 | .51 | .51 | .51 | .51 | .51 |
| 12 | .36 | .45 | .48 | .49 | .50 | .51 | .51 | .51 | .52 | .52 | .53 | .53 | .53 | .53 | .54 | .53 | .53 | .54 |
| 13 | .39 | .46 | .50 | .51 | .52 | .53 | .54 | .54 | .54 | .55 | .55 | .55 | .55 | .55 | .56 | .55 | .56 | .56 |
| 14 | .41 | .48 | .52 | .53 | .54 | .55 | .55 | .56 | .56 | .57 | .57 | .57 | .57 | .58 | .58 | .57 | .57 | .57 |
| 15 | .42 | .50 | .54 | .55 | .56 | .57 | .58 | .58 | .58 | .59 | .59 | .59 | .59 | .60 | .60 | .59 | .59 | .59 |
| 16 | .43 | .52 | .55 | .57 | .58 | .59 | .59 | .60 | .60 | .60 | .60 | .61 | .61 | .61 | .61 | .61 | .61 | .61 |
| 17 | .45 | .53 | .57 | .59 | .60 | .60 | .61 | .61 | .61 | .62 | .62 | .62 | .62 | .63 | .63 | .63 | .62 | .63 |
| 18 | .46 | .55 | .58 | .60 | .61 | .62 | .62 | .63 | .63 | .63 | .63 | .64 | .64 | .64 | .64 | .64 | .63 | .64 |
| 19 | .47 | .57 | .59 | .62 | .62 | .63 | .63 | .64 | .65 | .64 | .65 | .65 | .65 | .66 | .65 | .65 | .65 | .65 |
| 20 | .48 | .58 | .61 | .63 | .64 | .64 | .65 | .65 | .65 | .65 | .66 | .66 | .66 | .67 | .67 | .67 | .66 | .67 |
| 21 | .49 | .59 | .62 | .64 | .65 | .65 | .66 | .66 | .67 | .66 | .67 | .67 | .67 | .68 | .68 | .68 | .67 | .67 |
| 22 | .50 | .60 | .63 | .65 | .66 | .66 | .67 | .67 | .68 | .67 | .68 | .68 | .68 | .69 | .69 | .68 | .68 | .68 |
| 23 | .51 | .60 | .64 | .65 | .67 | .68 | .67 | .68 | .69 | .69 | .69 | .69 | .69 | .69 | .70 | .69 | .69 | .69 |
| 24 | .51 | .61 | .65 | .66 | .68 | .69 | .69 | .69 | .70 | .70 | .70 | .70 | .70 | .71 | .70 | .70 | .70 | .70 |
| 25 | .53 | .62 | .66 | .67 | .69 | .70 | .70 | .70 | .71 | .70 | .70 | .71 | .71 | .71 | .71 | .71 | .71 | .71 |
| 26 | .54 | .63 | .66 | .68 | .69 | .70 | .71 | .71 | .72 | .71 | .71 | .72 | .72 | .72 | .72 | .72 | .72 | .72 |
| 27 | .54 | .64 | .67 | .69 | .70 | .71 | .71 | .72 | .72 | .72 | .72 | .73 | .72 | .73 | .73 | .73 | .73 | .73 |
| 28 | .54 | .64 | .68 | .70 | .70 | .71 | .72 | .73 | .73 | .73 | .73 | .73 | .73 | .74 | .74 | .73 | .74 | .74 |
| 29 | .56 | .65 | .68 | .70 | .71 | .72 | .73 | .73 | .73 | .73 | .73 | .74 | .74 | .75 | .74 | .74 | .74 | .74 |
| 30 | .56 | .66 | .69 | .71 | .72 | .73 | .73 | .73 | .74 | .74 | .74 | .75 | .75 | .75 | .75 | .75 | .75 | .75 |
| 31 | .56 | .67 | .69 | .71 | .72 | .73 | .74 | .74 | .74 | .74 | .75 | .76 | .75 | .75 | .75 | .76 | .76 | .76 |
| 32 | .57 | .67 | .70 | .72 | .73 | .74 | .74 | .74 | .75 | .75 | .75 | .76 | .76 | .76 | .76 | .76 | .76 | .77 |
| 33 | .58 | .67 | .70 | .72 | .73 | .74 | .75 | .75 | .76 | .76 | .76 | .77 | .76 | .77 | .77 | .77 | .77 | .77 |
| 34 | .58 | .68 | .71 | .73 | .74 | .75 | .75 | .76 | .76 | .76 | .77 | .77 | .77 | .77 | .77 | .77 | .78 | .78 |
| 35 | .59 | .68 | .72 | .74 | .75 | .76 | .76 | .76 | .77 | .77 | .77 | .78 | .78 | .78 | .78 | .78 | .78 | .78 |
| 36 | .59 | .69 | .72 | .74 | .75 | .76 | .76 | .77 | .77 | .77 | .78 | .78 | .78 | .78 | .78 | .79 | .79 | .79 |
| 37 | .60 | .69 | .73 | .75 | .76 | .77 | .77 | .77 | .78 | .78 | .78 | .78 | .78 | .79 | .79 | .79 | .79 | .79 |
| 38 | .60 | .70 | .73 | .75 | .76 | .77 | .77 | .78 | .78 | .78 | .79 | .79 | .79 | .79 | .79 | .79 | .79 | .79 |
| 39 | .61 | .70 | .74 | .76 | .77 | .78 | .78 | .78 | .78 | .79 | .79 | .79 | .79 | .79 | .79 | .79 | .79 | .80 |
| 40 | .61 | .71 | .74 | .76 | .77 | .78 | .78 | .78 | .79 | .79 | .79 | .80 | .79 | .80 | .80 | .80 | .80 | .80 |

with probability. 95, but once this statement is made about one of the entries, the statistical accuracy of any other is not determinable without an involved conditional probability analysis. In Tables II and III we provide two heuristic checks on the estimated cyclical majority probabilities.

The *exact* probability calculated by Garman and Kamien [5] or by Sevcik [9] is shown to three decimal places in Table II.[3] The paucity of numbers in Table II is some indication of the difficulty of the problem. All the numbers for three issues are easily calculated with Garman and Kamien's formula but, other than those shown, there are no precisely known numbers for issues greater than three.

The sample 15,000 was generated in three separate groups of 5,000 each, and the probability estimates are available for each of the smaller samples. Some are shown in Table III. The smaller the variance among probability estimates over the replicated samples of 5,000 each, the more confidence there can be that the probabilities are generated by a stable process and that the numbers so obtained are good approximations to the true probabilities. The three sample estimates of any probability span a range less than or equal to .02 for 672 of the 684 numbers. For the twelve exceptions the range is .03. For example, the three

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[3] Garman and Kamien [5] and Sevcik [9] both counted the number of cycles, calculated the number $(I!)^J$, and then calculated their ratio. Sevcik made the calculation for $I = 3, 4, 5, 6, 7$ and $J = 3, 5, 7$. Garman and Kamien made these same calculations (plus others) except for $I = 6, 7$ and $J = 5, 7$.

TABLE II. EXACT PROBABILITY (TO 3 PLACES) OF A CYCLICAL MAJORITY*

| *Number of Issues* | *Number of Judges* | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 3 | 5 | 7 | 9 | 11 | ··· | 29 | ··· | ∞ |
| 3 | .056 | .069 | .075 | .078 | .080 | | .085 | | .088 |
| 4 | .111 | .139 | .150 | .156 | .160 | | | | |
| 5 | .160 | .200 | .215 | | | | | | |
| 6 | .202 | .251 | .271 | | | | | | |
| 7 | .239 | .296 | .318 | | | | | | |

* From Garman and Kamien [5] and Sevcik [9]. Numbers not shown are not known.

TABLE III. SELECTED PROBABILITY CALCULATIONS OF A CYCLICAL MAJORITY FROM THREE SAMPLES OF 5,000

| *Number of Issues* | *Number of Judges* | | | |
|---|---|---|---|---|
| | 5 | 15 | 25 | 35 |
| 10 | .39, .39, .39 | .47, .46, .46 | .47, .47, .47 | .49, .48, .47 |
| 20 | .59, .57, .57 | .66, .64, .64 | .66, .65, .67 | .67, .66, .65 |
| 30 | .67, .67, .64 | .73, .74, .73 | .74, .75, .76 | .75, .75, .75 |
| 40 | .61, .60, .61 | .77, .79, .79 | .79, .80, .80 | .80, .81, .79 |

estimates for 15 issues and 3 judges were .42, .43, and .40. In the data that generate Table I there are 105 entries identical in all three samples, 453 entries where two of the estimates are alike, and 114 entries where each of the three samples gives a different estimate.

Our simulation provides several times as many cyclical majority probability estimates as any previous simulation. The cost of finding so many numbers is our inability to place statistical confidence limits on any but one of them. The reader can see that our estimates are never more than .01 away from the true probabilities when such are known; we think it reasonable to conclude that the others are, at worst, only slightly less accurate. Think of it this way: the estimate shown for 40 issues and 37 judges is known to be accurate within .01 with probability .95. The nature of the process is such that if there are systematic biases the greatest inaccuracy will be for 3 issues and 3 judges. But the benchmark comparisons indicate that the estimates for 3 issues and 3 judges is also accurate to .01.

## The Method

We now give our modus operandi for generating sample voting patterns and determining whether there is a majority winner or a cyclical majority. We reformulated the problem for efficient computer simulation.

For conceptual convenience we first describe the procedure as it appears for a fixed number of issues and judges, and then we show how the evaluation is done simultaneously for any lesser number of judges or issues or both. Given a fixed number $m$ of judges and fixed number $n$ of issues, our computer program generates (15,000) "outcomes," defined as a random vote pattern for each judge and a count to determine the existence of a majority winner. For each outcome, all judges but one are allowed to state their preferences at random. (The first judge always states his preferences in a fixed, canonical order—a mere labeling convenience.) After all the preference orderings are known, each issue is tested as a possible winner. If no winner is found, a cycle is indicated.

The preference orderings are developed in a *preference vector*. The testing for a winner is accomplished by developing an overall *preference matrix* (first used by Garman and Kamien [5]) for each judge and then overall for the outcome. The entry in row $r$ and column $c$ of the matrix for a specific judge is 1 if issue $c$ is preferred to issue $r$ and 0 otherwise. The overall preference matrix is the sum of the individual preference matrices. The criterion used is that a majority winner exists if and only if there is a row in the overall preference matrix all of whose entries are less than $m/2$ ($m$ = the number of judges). Two examples are shown in Figure 1, each with three judges ($m = 3$), $A$, $B$, and $C$, and four issues ($n = 4$), $i$, $j$, $k$, and $l$.

The preference order of the first judge, $A$, is fixed and the row and column labels for the overall preference matrix are chosen the same as in $A$'s preference matrix. As each judge's preference vector becomes known, his individual preference matrix is generated and added to the overall preference matrix. After matrices have been generated for all judges and summed to form the overall matrix, each row of the overall matrix is scanned for an entry greater than $m/2$ (1.5 in the examples). If such an entry is found, the issue represented by that row cannot be a majority winner because there is some other issue that at least half the judges prefer. If some row has all its entries less than $m/2$, it is the majority winner's row. Otherwise, each row has at least one entry greater than $m/2$; hence there is no clear majority winner and a cyclic majority has occurred.

In the first example, issue $k$ is the majority winner and the overall preference matrix elements in the $i$ and $j$ rows which caused rejection of these rows are circled. In the second example, no majority winner exists and the elements that forced each issue's rejection are circled.

Observation of our first efforts to program vote sampling showed that most of the computation time was spent in generating random preference vectors. For $m$ judges and $n$ issues, the vote generating process is proportional to $mn^2$, while the evaluating process is proportional only to $n^2$. We concluded that for a small fixed increment in processing time, the program could report, for each preference vector, not only the results for $m$ judges and $n$ issues, but also other results for any combination of judges up to $m$ (odd) and issues up to $n$.

Recall that a preference matrix is developed for each judge and then a sum over all judges is computed. Before summing we have $m$ preference matrices, one for each judge, over $n$ issues. For any number $m' < m$ judges and $n' < n$ issues we can obtain an overall preference matrix using only the first $m'$ judges and the first $n'$ issues. The overall preference matrix constructed for $m'$ judges and $n'$ issues can itself be tested for a cyclical majority. Thus a given set of preferences for $m$ judges over $n$ issues can be used to generate a sample voting pattern for all possible judge–issue pairs $(m', n')$ as long as $m' \leq m$ and $n' \leq n$. The outcomes of the sample votes over various values of $m'$ and $n'$ are not independent of the outcome for $m$ and $n$, but neither are they dependent in any known way. We believe it fair to say they are dependent in the same sense as outputs from a pseudo-random number generator are dependent. In general, a large body of data is reduced to one bit of information—a cyclic majority has or has not occurred. Reducing the number of judges or issues may change a cyclic majority into an outcome with a majority winner, or vice versa.

The actual details of the machine language program for efficiently manipulating bits to make the proper counts are tedious and, for lack of general application, are not suitable for presentation here. The program and details will be sent to interested readers on request.

| *(Ascending) preference order for each judge over the issues* | *1st Example* | *2nd Example* |
|---|---|---|
| Judge *A* | $i < j < k < l$ | $i < j < k < l$ |
| *B* | $l < j < i < k$ | $l < i < j < k$ |
| *C* | $j < l < i < k$ | $j < k < l < i$ |

*Preference matrices for each judge: A "1" in row r and column c indicates that for this judge, the issue represented in column c is preferred to that in row r.*

|  |  | $i$ | $j$ | $k$ | $l$ | $i$ | $j$ | $k$ | $l$ |
|---|---|---|---|---|---|---|---|---|---|
| *A* | $i$ | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 |
|  | $j$ | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
|  | $k$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
|  | $l$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| *B* | $i$ | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 |
|  | $j$ | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
|  | $k$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
|  | $l$ | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |
| *C* | $i$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
|  | $j$ | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 |
|  | $k$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
|  | $l$ | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 |

*Sum of preference matrices: (Circled elements eliminate that row issue as majority winner; row with * represents majority winner).*

|  |  |  |  |  |  |  |  |  |
|---|---|---|---|---|---|---|---|---|
| $i$ | 0 | 1 | ③ | 1 | 0 | ② | 2 | 1 |
| $j$ | ② | 0 | 3 | 2 | 1 | 0 | ③ | 2 |
| *$k$ | 0 | 0 | 0 | 1 | 1 | 0 | 0 | ② |
| $l$ | 2 | 1 | 2 | 0 | ② | 1 | 1 | 0 |
|  | issue $k$ wins |  |  |  | no winner |  |  |  |

FIG. 1. Two examples of program's operation (3 judges, 4 issues)

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